Homework 3 – Sketch of Solutions

#1 With base points omitted, let $\tilde{\theta} : \pi(\tilde{G}) \oplus \pi(\tilde{G}) \longrightarrow \pi(\tilde{G} \times \tilde{G})$ be the standard isomorphism. Then

$$\mu_*(p \times p)_* \tilde{\theta}(\alpha, \beta) = p_*(\alpha)\, p_*(\beta) \in p_* \pi(\tilde{G}).$$

$\therefore$ $(\mu(p \times p))_* \pi(\tilde{G} \times \tilde{G}) \subseteq p_* \pi(\tilde{G})$ so there is a lift $\tilde{\mu}$

$$\begin{tikzcd}
\tilde{G} \times \tilde{G} \arrow[r, "\tilde{\mu}"] \arrow[d, "p \times p"] & \tilde{G} \arrow[d, "p"] \\
G \times G \arrow[r, "\mu"] & G
\end{tikzcd}$$

But $\tilde{G} \xrightarrow{i_1} \tilde{G} \times \tilde{G} \xrightarrow{\tilde{\mu}} \tilde{G}$ and $id : \tilde{G} \longrightarrow \tilde{G}$ are both lifts of $p : \tilde{G} \to G$. $\therefore$ $\tilde{\mu} i_1 = id$. Similarly $\tilde{\mu} i_2 = id$. Also $\tilde{\mu}(\tilde{\mu} \times id)$, $\tilde{\mu}(id \times \tilde{\mu}) : \tilde{G} \times \tilde{G} \times \tilde{G} \longrightarrow \tilde{G}$ are both lifts of $\mu(\mu \times id)(p \times p \times p) = \mu(id \times \mu)(p \times p \times p)$. $\therefore$ Associativity

#2 Let $i : G \to G$ be the inverse map ($i(g) = g^{-1}$). Then show $\tilde{G} \xrightarrow{p} G \xrightarrow{i} G$ lifts to a map $\tilde{i} : \tilde{G} \to \tilde{G}$. The condition for $\tilde{i}$ to be an inverse is $\tilde{\mu}(id \times \tilde{i})\Delta = c_{\tilde{e}}$, where $\Delta : \tilde{G} \to \tilde{G} \times \tilde{G}$ is defined by $\Delta(x) = (x, x)$. Show that $\tilde{\mu}(id \times \tilde{i})\Delta$ and $c_{\tilde{e}}$ are lifts of the same map. Next $K = \text{kernel } p = p^{-1}(e)$ and so is discrete and also normal. Let $\beta \in K$ and define $f : G \to K$ by $f(\alpha) = \alpha \beta \alpha^{-1}$. Since G is connected and K is discrete, $f(G)$ is a point. $\therefore$ $f(G) = \beta$ so $\alpha\beta = \beta\alpha$. $\therefore$ $\beta \in$ center.

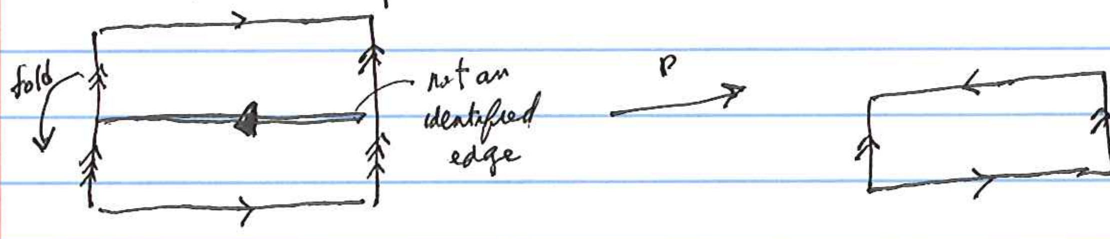
#5 Define $\theta : \pi(X, x_0) \to p^{-1}(x_0)$ by $\theta[\ell] = \tilde{\ell}(1)$ where $\tilde{\ell}$ is a lift of ℓ which starts at $\tilde{x}_0$. Then show θ is onto. Since X is simply connected, $p^{-1}(x_0)$ is a single point. $\therefore$ $p : \hat{X} \to X$ is one-one. But it is open, continuous and onto (Problem #4) $\therefore$ p is a homeo.

#6 $f \sim p_n$ for some n, where $p_n z = z^n$. $\therefore$ $\deg f = \deg p_n = n$. But

$f_*(\alpha) = p_{n*}(\alpha) = n\alpha = (\deg f)\alpha$ (reason $p_{n*}(\alpha) = n\alpha$: let $\alpha = [a]$, $p_n a = a^n = na$ (additive notation)

#7 $f: X \to S^1$ $\quad f_*\pi(X)$ finite $\subseteq \pi(S^1) = \mathbb{Z}$ $\quad \therefore f_*\pi(X) = 0 \subseteq p_*\pi(\mathbb{R})$ ($p: \mathbb{R} \to S^1$ cover) $\therefore f$ lifts to $\mathbb{R}$: $f = pf'$, where $f': X \to \mathbb{R}$. But $\mathbb{R}$ is contractible, so $id \simeq c_0: \mathbb{R} \to \mathbb{R}$.
$\therefore f = p(id)f' \simeq p c_0 f' = c$.

#8



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#9 $\theta: p^{-1}(x_0) \longrightarrow p^{-1}(x_1)$ defined by $\theta(\tilde{x}_0) = \tilde{l}(1)$. $\mu: p^{-1}(x_1) \to p^{-1}(x_0)$ defined by $\mu(\tilde{x}_1) = \tilde{l}(1)$. Show $\theta\mu = id$, $\mu\theta = id$.

#10 (a)

$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d, "p_2"', "\text{quotient map}"] & S^1 \arrow[d, "p_2"] \\ S^1 \arrow[ur, dashed, "\hat{g}"] \arrow[r, dashed, "g"] & S^1 \end{tikzcd}$$

$p_2 f$ ~~induces~~ induces g, $gp_2 = p_2 f$.
$(\deg g)(\deg p_2) = \deg(gp_2) = \deg(p_2 f) = (\deg p_2)(\deg f)$ $\quad \therefore \deg f = \deg g$

(b) Suppose $\deg f = 2k$, $k > 0$. $\quad 2g_*\pi(S^1) = g_* p_{2*}\pi(S^1) = p_{2*} f_* \pi(S^1) = 2k\, p_{2*}\pi(S^1)$ $\quad$ so $g_*\pi(S^1) = k\, p_{2*}\pi(S^1) \subseteq p_{2*}\pi(S^1)$ $\quad \therefore g$ lifts to $\hat{g}$: $p_2\hat{g} = g$.

(c) $p_2\hat{g}p_2 = p_2 f$, $\quad p_2\hat{g}p_2(1,0) = p_2 f(1,0)$ $\quad \therefore \hat{g}p_2(1,0) = \pm f(1,0)$.
If $+$, $\hat{g}p_2 = f$. If $-$, $f(-1,0) = -f(1,0) = \hat{g}p_2(1,0) = \hat{g}p_2(-1,0)$.
so $\hat{g}p_2 = f$. But $\hat{g}p_2$ is an even function and f is an odd function. Contradiction. $\therefore \deg f$ is odd.

#11 Let $f: S^1 \to S^1$ be even function $f(z) = f(-z)$ so f induces f'

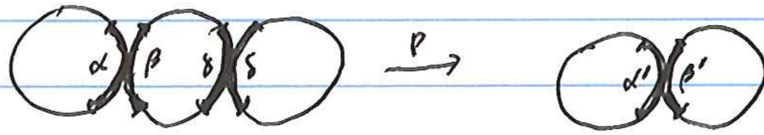
$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d, "p_2"'] & S^1 \\ S^1 \arrow[ur, dashed, "f'"'] & \end{tikzcd}$$

since p_2 is a quotient map.
$\deg f = \deg p_2 \deg f' = 2 \deg f'$

#12 $\{U\}$ elementary nbhs of X. $p^{-1}(U) = \bigcup V_j$. Claim $\{U \cap A\}$ elementary nbhs of A $\quad p^{-1}(A \cap U) = \tilde{A} \cap \bigcup V_j = \bigcup \tilde{A} \cap V_j$

$p|_{\tilde{A} \cap V_j} : \tilde{A} \cap V_j \xrightarrow{\approx} A \cap U$ homeo.

#13



$p : \alpha \to \alpha'$, $\quad p : \beta \to \beta'$, $\quad p : \gamma \to \beta'$, $\quad p : \delta \to \alpha'$

#14 Map $F(\alpha, \beta) \longrightarrow F(\gamma, \delta)$ by λ

$\lambda(\alpha) = \alpha^{-1}\beta^{-1} = \delta$

$\lambda(\beta) = \beta = \gamma$ $\qquad$ and

$F(\gamma, \delta) \longrightarrow F(\alpha, \beta)$ by ρ

$\rho(\gamma) = \gamma = \beta$

$\rho(\delta) = \gamma\delta^{-1} = \alpha$